Encapsulated Petri Nets

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[1] Pauline N. Kawamoto, Yasushi Fuwa, Yatsuka Nakamura.
Basic Concepts for Petri Nets with Boolean Markings.
Journal of Formalized Mathematics, 5, 1993.
http://www.mizar.org
[2] Waldemar Korczynski. Definitions of Petri Net. Part III.
Journal of Formalized Mathematics, 4, 1992.
http://mizar.org
[3] Antoni A. Urban. Binary Arithmetic on *B*-bit Numbers.
http://www.softm.co.za
[4] Edmund Woronowicz. Many-Argument Relations.
Journal of Formalized Mathematics, 2, 1990.
http://www.mizar.org

Abstract. Petri Nets concept is well known and widely used. Encapsulated Petri Nets, if applied, could change computer software drasticly. To be able to define extended Petri Nets and then encapsulated Petri Nets, the basic definition of the concept had to be simplified. The new definition is equivalent to the classic one, but very clear and compact. Then transitions are associated with functions, procedures, and finally, with other Petri Nets. The addition of numbers is given as a sample. If Encapsulated Petri Nets can add numbers, they will do everything.

The following notation is used in this article:
• $S, U, V, W, D, M, N, X, Y, Z$ denote sets;
• s, x, y, z denote elements of the sets;
• T, Q, R denote relations;
• $f, g, F, G, \mathcal{F}, \Phi, \Psi$ denote functions;
• $\mathcal{P}$ denotes structure;
• $<a, b>$ is a tuple, i.e. an ordered pair of a and b;
• $A \times B$ is a Cartesian product of the sets A and B.

1. New Definition of Petri Nets

DEFINITION 1. Let S be a set (it can be empty). We denote by 2^S the set of all subsets of S. Let us assume that T is a binary relation on 2^S, i.e. $\emptyset \subseteq T \subseteq 2^S \times 2^S$. Structure $\mathcal{P} = <S, T>$ is called *Petri Net*. Elements of set S are called *places*, elements of relation T are called *transitions*. The definition is equivalent to the classic one [1].

Having formulated so simple definition of Petri Nets, we can apply all known tools of relations testing. While doing this we must remember that elements of T are tuples of subsets of S. For all $t \in T$ there exist $U, V \subseteq S$ such that $t = <U, V>$.

We can apply the following definitions [4]:
• $\text{Domain}_T = \{ U \in 2^S : \exists V \in 2^S \text{ such that } <U,V> \in T \}$
• $\text{Range}_T = \{ V \in 2^S : \exists U \in 2^S \text{ such that } <U,V> \in T \}$
• $\text{Field}_T = \text{Domain}_T \cup \text{Range}_T$
• $T^{-1} = \{ <V,U> \in 2^S \times 2^S : <U,V> \in T \}$
• $T_1 \bullet T_2 = \{ <U,W> \in 2^S \times 2^S : \exists V \in 2^S \text{ such that } <U,V> \in T_1 \text{ and } <V,W> \in T_2 \}$
• If $Q \subset 2^S$ then $T_{|Q} = \{ <U,V> \in 2^S \times 2^S : U \in Q \}$
• If $R \subset 2^S$ then $_{R|}T = \{ <U,V> \in 2^S \times 2^S : V \in R \}$
but in this particular case we should look at elements of S rather than 2^S. For example, we define domain, range and field of T as follows:
• $\text{Dom}_T = \{ s \in S : \exists U \in 2^S \text{ such that } s \in U \text{ and } <U,V> \in T \}$
• $\text{Rng}_T = \{ s \in S : \exists V \in 2^S \text{ such that } s \in V \text{ and } <U,V> \in T \}$
• $\text{Fld}_T = \text{Dom}_T \cup \text{Rng}_T$
We know that Fld_T is a subset of S, therefore, we can eliminate all isolated places of the initial set S.

We cannot say if the relation T is
• reflexive in 2^S, i.e. $\forall U \in 2^S : <U,U> \in T$;
• irreflexive in 2^S, i.e. $\forall U \in 2^S : <U,U> \notin T$;
• symmetric in 2^S, i.e. $\forall U,V \in 2^S : <U,V> \in T \Rightarrow <V,U> \in T$;
• antisymmetric in 2^S, i.e. $\forall U,V \in 2^S : (<U,V> \in T \text{ and } <V,U> \in T) \Rightarrow U=V$;
• asymmetric in 2^S, i.e. $\forall U,V \in 2^S : <U,V> \in T \Rightarrow <V,U> \notin T$;
• connected in 2^S, i.e. $\forall U,V \in 2^S : U \neq V \Rightarrow (<U,V> \in T \text{ or } <V,U> \in T)$;
• strongly connected in 2^S, i.e. $\forall U,V \in 2^S : <U,V> \in T \text{ or } <V,U> \in T$;
• transitive in 2^S, i.e. $\forall U,V,W \in 2^S : (<U,V> \in T \text{ and } <V,W> \in T) \Rightarrow <U,W> \in T$;
We even do not want T to meet one of the above-listed characteristics - with exception of the irreflexiveness. To test T we must once again go from 2^S to S.

DEFINITION 2. Let two places $s_0, s \in S$ be given. We say that $<s_0,s>$ *is an arc* iff

$$\exists <U,V> \in T \text{ such that } s_0 \in U \text{ and } s \in V \tag{1}$$

We denote by R the set of all arcs. R is a binary relation on S.

DEFINITION 3. Let two places $s_0, s \in S$ be given. We say that place s *is reachable from place* s_0 iff $\exists k \in \mathbf{N}$ and $\exists s_1, \ldots, s_k \in S$ such that

$$s_k = s \text{ and } \bigvee_{i=1}^{k} <s_{i-1}, s_i> \in R \tag{2}$$

We denote by Q the set of all tuples $<s_0,s>$ such that s is reachable from s_0. It is a binary realtion on S.

We now can check if relation Q is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, connected, strongly connected, or transitive in S. Moreover, characteristics of Q can be associated with T. We say that:
• T is reflexive in S iff Q is reflexive in S;
• T is irreflexive in S iff Q is irreflexive in S;
• T is symmetric in S iff Q is symmetric in S;
• T is antisymmetric in S iff Q is antisymmetric in S;
• T is asymmetric in S iff Q is asymmetric in S;

• T is connected in S iff Q is connected in S;
• T is strongly connected in S iff Q is strongly connected in S;
• T is transitive in S iff Q is transitive in S;

Remark. Please note that in the definitions above we are not using the phrase "in 2^S" but the phrase "in S".

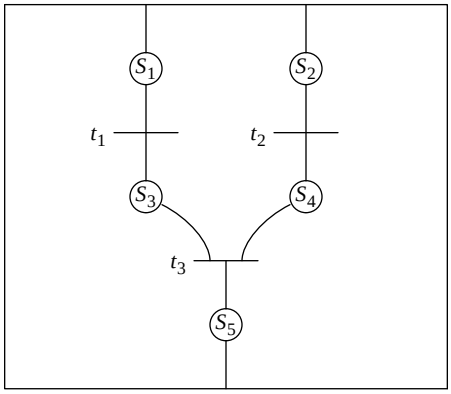
2. Firing and Running

Notation. Let us take a subset M of S, $\emptyset \subseteq M \subseteq S$. We denote by $T_M = \{ <U,V> \in T : \emptyset \subset U \subseteq M \}$ the set of so called *firable transitions*. Additionally, we define two subsets of places:

$$T_M^- = \bigcup_{<U,V> \in T_M} U \qquad T_M^+ = \bigcup_{<U,V> \in T_M} V \tag{3}$$

We also define $\mathrm{Init}_T = \{ <U,V> \in T : U = \emptyset \}$ and $\mathrm{End}_T = \{ <U,V> \in T : V = \emptyset \}$ as well as two subsets of places:

$$S^I = \bigcup_{<U,V> \in \mathrm{Init}_T} U \qquad S^E = \bigcup_{<U,V> \in \mathrm{End}_T} V \tag{4}$$



Sample. The picture shows a very simple Petri Net:

$S = \{ s_1, s_2, s_3, s_4, s_5 \}$;

$T = \{ <\emptyset, \{ s_1, s_2 \}>, <\{ s_1 \}, \{ s_3 \}>,$
$<\{ s_2 \}, \{ s_4 \}>, <\{ s_3, s_4 \}, \{ s_5 \}>,$
$<\{ s_5 \}, \emptyset> \}$;

$S^I = \{ s_1, s_2 \}$;

$S^E = \{ s_5 \}$;

If $M = \{ s_1 \}$ then $T_M^- = \emptyset$, $T_M^+ = \{ s_3 \}$.

DEFINITION 4. We now define functor $\Phi : 2^S \to 2^S$ called *firing*. Let M be a subset of S. We denote by N the value $\Phi(N)$ and calculate N as follows:
1^0 If $M = \emptyset$ then $N = S^I$ where S^I is defined in (4).
2^0 If $\emptyset \subset M \subseteq S$ then $N = M \setminus T_M^- \cup T_M^+$ where T_M^- and T_M^+ are defined in (3).
Remark. If we look at definition 4 we shall see clear steps of firing. Firstly, we select all firable transitions (T_M). Secondly, we subtract T_M^- from M, i.e. we change so called Boolean markings of all places followaed by firable transitions. Thirdly, we fire all transitions T_M simultaneously (at the moment we do nothing). And finally, we add T_M^+ to M, i.e. we change Boolean markings of all places following the transitions T_M.

DEFINITION 5. The procedure of *running* Petri Net $\mathcal{P} = <S,T>$ is performed in the following steps:
1. $M := \Phi(\emptyset) = S^I$
2. $M := \Phi(M)$ as it is described in definition 4
3. We repeat step 2 until $M = \emptyset$

DEFINITION 6. Let two Petri Nets $\mathcal{P}_1 = <S_1, T_1>$ and $\mathcal{P}_2 = <S_2, T_2>$ be given and one transition $<U_1, V_1> \in T_1$ be selected. We assume that:

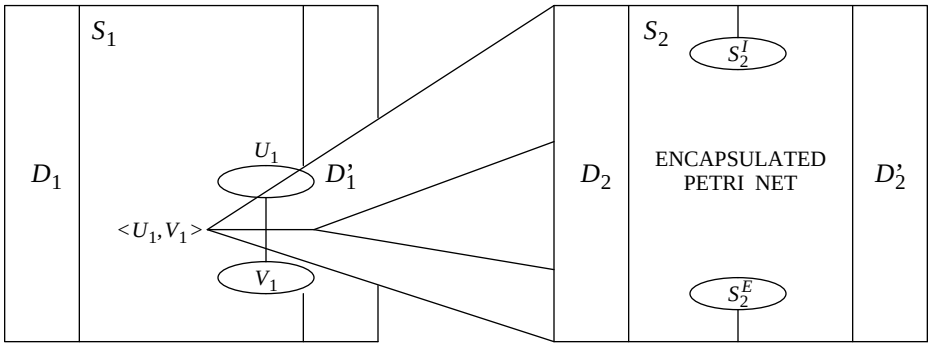
$$S_1 \cap S_2 = \emptyset \text{ and } \text{Init}_{T_2} = \{ <\emptyset, S_2^I> \} \text{ and } \text{End}_{T_2} = \{ <S_2^E, \emptyset> \} \tag{5}$$

We now define a new Petri Net $\mathcal{P} = <S, T>$ using the following formulae:

$$\begin{gathered} S = S_1 \cup S_2 \text{ and} \\ T = (T_1 \setminus \{ <U_1, V_1> \}) \cup (T_2 \setminus \text{Init}_{T_2} \setminus \text{End}_{T_2}) \cup \{ <U_1, S_2^I>, <S_2^E, V_1> \} \end{gathered} \tag{6}$$

We say that the second Petri Net was *included* to the first one in $<U_1, V_1> \in T_1$.

Picture.

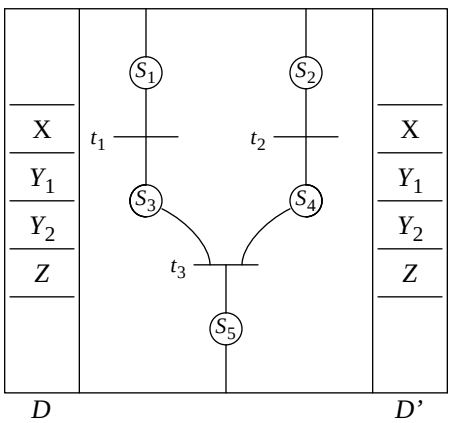


3. Extended Petri Nets

DEFINITION 7. Let us assume that $\mathcal{P} = <S, T>$ is a Petri Net. We now consider non-empty sets $X_1, \ldots, X_K$. We denote by D and call *domain* the Cartesian product of the considered sets, i.e.

$$D = \bigtimes_{i=1}^{K} X_i \tag{7}$$

Then, we define functor Ψ from the set T to the set $\mathcal{F}$ of all functions $f: D \to D$. A new structure $<S, T, D, \Psi>$ is called *extended Petri Net* and denoted by $\mathcal{P}_+$.



Sample. We consider the same Petri Net as previously. On the sides of the diagram we place domain D and its mirror image D'. Let D be equal to $X \times Y_1 \times Y_2 \times Z$. Then we build functor Ψ yielding transitions T and ranging over set $\mathcal{F}$.
$\Psi(t_1) = f$ such that $y_1 = f(x)$
$\Psi(t_2) = g$ such that $y_2 = g(x)$
$\Psi(t_3) = h$ such that $z = h(y_1, y_2)$
For all other transitions the value of Ψ is equal to id_D. In this sample extended Petri Net performs calculation:
$z = h(f(x), g(x))$

Notation. If $f:A\times B\to A\times B$ is a function then $f=(f_A,f_B)$ where $f_A:A\times B\to A$ and $f_B:A\times B\to B$. For all $a\in A$ and $b\in B$ holds $f(a,b)=(f_A(a,b),f_B(a,b))$.

DEFINITION 8. Let $\mathcal{P}_+=<S,T,D,\Psi>$ be given. We now consider domain D defined as follows. There exists non-empty subset S^* of S such that:

$$D=X\times 2^{S^*} \text{ where } X=\bigtimes_{i=1}^{K}X_i \tag{8}$$

We now define $\mathcal{F}=\{f:D\to D:$ for all $x\in X$ and $A\subseteq S^*$ holds $A\subseteq f_{2^{S^*}}(x,A)\}$. The structure $\mathcal{P}_+=<S,T,D,\Psi>$ is called *extended Petri Net with overlapping in* S^*. To complete the definition we have to define functor Φ of firing. Let M be a subset of S and $x\in X$. We define the following notation:

$$S_M^*=\bigcup_{t\in T_M}(\Psi(t))_{2^{S^*}}(x,M\cap S^*) \tag{9}$$

We denotote by N the value of $\Phi(M)$ and calculate N as follows:
1^0 If $M=\emptyset$ then $N=S_M^*\cup S^I$ where S^I is defined in (4) and S_M^* is defined in (9).
2^0 If $\emptyset\subset M\subseteq S$ then $N=M\setminus T_M^-\cup S_M^*\cup T_M^+$ where T_M^- and T_M^- are defined in (3) and S_M^* is defined in (9).

DEFINITION 9. Let two extended Petri Nets $\mathcal{P}_1=<S_1,T_1,D_1,\Psi_1>$ with overlapping in S_1^* and $\mathcal{P}_2=<S_2,T_2,D_2,\Psi_2>$ with overlapping in S_2^* - be given and one transition $<U_1,V_1>\in T_1$ be selected. We assume that:

$$\begin{gathered}S_1\cap S_2=\emptyset,\ \mathrm{Init}_{T_2}=\{<\emptyset,S_2^I>\},\ \mathrm{End}_{T_2}=\{<S_2^E,\emptyset>\}\\ \text{and }\ \Psi_1(U_1,V_1)=\mathrm{id}_{D_1\times S_1^*}\end{gathered} \tag{10}$$

We now define a new Petri Net $\mathcal{P}=<S,T,D,\Psi>$ using the following formulae:

$$\begin{gathered}S=S_1\cup S_2,\ S^*=S_1^*\cup S_2^* \text{ and } D=D_1\times(D_2\setminus D_1)\\ T=(T_1\setminus\{<U_1,V_1>\})\cup(T_2\setminus\mathrm{Init}_{T_2}\setminus\mathrm{End}_{T_2})\cup\{<U_1,S_2^I>,<S_2^E,V_1>\}\\ \Psi(U_1,S_2^I)=\Psi_2(\mathrm{Init}_{T_2}) \text{ and } \Psi(S_2^E,U_1)=\Psi_2(\mathrm{End}_{T_2})\end{gathered} \tag{11}$$

We say that the second Petri Net was *included* to the first one in $<U_1,V_1>\in T_1$.

Sample. Addition of numbers as extended Petri Net with overlapping.
$S=\{s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8,s_9,s_{10}\}$
$T=\{<\emptyset,\{s_1\}>,<\{s_1\},\{s_2\}>,<\{s_2\},\{s_3,s_4\}>,<\{s_3\},\{s_5\}>,<\{s_4\},\{s_6\}>,$
$<\{s_5,s_6\},\{s_7\}>,<\{s_5,s_6\},\{s_8\}>,<\{s_7\},\{s_9\}>,<\{s_8,s_9\},\{s_{10}\}>,<\{s_{10}\},\emptyset>\}$
$D=\{<X,Y,Z,x,y,z,c,i>\}\times 2^{\{s_2\}}$
$\Psi(\{s_1\},\{s_2\})$: $c:=0,\ i:=0$
$\Psi(\{s_3\},\{s_5\})$: $x:=\mathrm{GetBit}(i,X)$
$\Psi(\{s_4\},\{s_6\})$: $y:=\mathrm{GetBit}(i,Y)$
$\Psi(\{s_5,s_6\},\{s_7\})$: $z:=F(x,y,c)$ [3]
$\Psi(\{s_5,s_6\},\{s_8\})$: $c:=G(x,y,c)$ [3]
$\Psi(\{s_7\},\{s_9\})$: $Z:=\mathrm{PutBit}(i,z)$
$\Psi(\{s_8\},\{s_9\})$: $<i,s_2>:=\mathrm{Next}(i)$; if $i<I$ then add $\{s_2\}$ to the current M.

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