Binary Arithmetic on *B*-bit Numbers

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Summary. Tarski Grothendieck axiom states that "everything is a set". Kuratowski Zorn lemma helps to overcome cardinality problems. This article presents definitions of a collection of bits, *B*-bit numbers, set-like binary representation. The last definition is then compared to the standard binary representation. The fundamentals of Binary Arithmetic are presented, but all operations on bits are regaded as sets. One-placed and two-placed operations on bits as well as their compositions are given as samples. Several propositions, regarding operations as sets, are listed. Finally, one-placed and two-placed bitwise operations on *B*-bit numbers are explained. As a result of the discussion in this paper, we can say that the addition on numbers is an **8-element** set. Who said that numbers are a result of counting? How to count if we have no numbers? Numbers are well ordered pieces of a universal set. Moreover, they are probably well ordered by time.

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The concept of this article is based mainly on: [6] by A. Trybulec, [8] by W. Trybulec and G. Bancerek, [4] by T. Nishiyama and Y. Mizuhara, [2] and [3] by C. Bylinski, as well as [7] by A. Trybulec. Many thanks to the authors for such valuable inspiration. The articles [10], [5], [1], [9] and [11] provide notation and terminology for this paper, and were also very helpful.

We shall use the following symbols and special notation:
• $\mathcal{U}$ is a universal set for all considerations of this article;
• *A, B, P, N, D, T, F, G, H, X, Y, Z* as well as *a, b, p, n, d, t, x, y, z* denote sets;
• *f, g, h, φ, ψ, Φ, ω* are functions, operations or functors;
• *R, Q* are relations;
• Symbol $:=$ denotes that we assign value to a variable. For example, let f: **N** $\to$ **N** the function $f(x) = x + 1$. Notation $x := x + 1$ means that $x \to f(x) = x + 1$.

A very important role is played by the quoted below scheme.

> The scheme *Zorn Max* deals with a non empty set $\mathcal{A}$ and states that:
> There exists an element x of $\mathcal{A}$ such that for every element y of $\mathcal{A}$ such that $x \neq y$ holds not $\mathcal{P}[x,y]$, provided the following conditions are met:
> • For every element $x \in \mathcal{A}$ holds $\mathcal{P}[x,x]$;
> • For all elements x, y of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,x]$ holds $x = y$;

- For all elements x, y, z of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[y,z]$ holds $\mathcal{P}[x,z]$;
- Let X be given. Suppose $X \subseteq \mathcal{A}$ and for all elements x, y of $\mathcal{A}$ such that $x \in X$ and $y \in X$ holds $\mathcal{P}[x,y]$ or $\mathcal{P}[y,x]$. Then there exists an element y of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ such that $x \in X$ holds $\mathcal{P}[x,y]$.

1. SET-LIKE BINARY REPRESENTATION

Suppose $\mathcal{U}$ is a non empty universal set, B is a non empty subset of $2^{\mathcal{U}}$, R is a relation on B.

DEFINITION 1. Stucture $<B,R>$ is said to be a *collection of bits* iff:

(1) R is well ordering B;

(2) $\forall \emptyset \subset A \subseteq B \ \exists a \in A$ such that $\forall b \in A \setminus \{a\}$ holds $<b,a> \notin R$;

(3) $\forall \emptyset \subseteq A \subseteq B \ (\bigcap_{a \in A} a) \cap (\bigcap_{b \in B \setminus A} \mathcal{U} \setminus b) \neq \emptyset$

PROPOSITION

(4) Let A be a non empty subset of B. We define $Q = R \cap [:A,A:]$. Then $<A,Q>$ is a collection of bits.

DEFINITION 2. Let us consider the collection of bits $<B,R>$. Let A be a subset of B such that $\emptyset \subseteq A \subseteq B$. We can also say that $A \in 2^B$. The subset A is called *set-like binary representation*.

We now define the set P as follows:

DEFINITION 3.

(5) $P = \{ (\bigcap_{a \in A} a) \cap (\bigcap_{b \in B \setminus A} \mathcal{U} \setminus b) :$ for all A such that $\emptyset \subseteq A \subseteq B \}$

is called a set of *B-bit numbers*.

REMARK. There is still no order on P. But, the relation R on B implies a relation Q on P. Let $x \in P$ and $y \in P$. In accordance with (5):

(6) $\exists X \subseteq B$ such that $x = (\bigcap_{a \in X} a) \cap (\bigcap_{b \in B \setminus X} \mathcal{U} \setminus b)$, and

(7) $\exists Y \subseteq B$ such that $y = (\bigcap_{a \in Y} a) \cap (\bigcap_{b \in B \setminus Y} \mathcal{U} \setminus b)$.

We define Q as follows:

(8) $<x,y> \in Q$ iff $(X = Y)$ or $(\exists d \in Y \setminus X$ such that $\forall c \in X \setminus Y$ holds $<d,c> \in R)$.

PROPOSITION.

(9) The relation Q on P defined by (8) well orders P.

2. STANDARD BINARY REPRESENTATION

DEFINITION 4. Let a non empty set T be given. We define a set S called *strings of elements of T* recursively:

(10) $\forall t \in T$ holds $t \in S$ and $t \in S$

(11) $\forall s \in S$ and $\forall t \in T$ holds $ts \in S$

Definition 4 will allow us to formalize *standard binary representation*.

DEFINITION 5. Let us take a set $D = \{ \circ, \mathbb{I} \}$ called *digits* and the set *S* of strings of digits. Suppose *p* is a B-bit number, i.e. belongs to *P* defined in (5). It means that there exists $A \subseteq B$ such that:

(12) $p = (\bigcap_{a \in A} a) \cap (\bigcap_{b \in B \setminus A} \mathcal{U} \setminus b)$

(13) According to (2) there exists $b \in B$ such that $\forall a \in B \setminus \{ b \}$ holds $< b, a > \notin R$.

(14) If $b \in A$ then $s := \mathbb{I}$, but if $b \notin A$ then $s := \circ$.

(15) We now change set *B* subtracting element *b*, we denote $B := B \setminus \{ b \}$.

(16) According to (2) there exists $b \in B$ such that $\forall a \in B \setminus \{ b \}$ holds $< b, a > \notin R$.

(17) If $b \in A$ then $s := \mathbb{I}s$, but if $b \notin B$ then $s := \circ s$.

Please note that the above-written definition is recursive. Therefore, steps (15), (16) and (17) must be repeated.

Definition 5 creates a function *f* with domain *P* and codomain *S*. Let us consider

(18) $j : P \to \operatorname{rng} f \subset S$, where for all $p \in P$ holds $j(p) = f(p)$

PROPOSITION. The function *j* defined in (18) and created by definition 5 is a bijection.

DEFINITION 6. For all $p \in P$ its image under *j*, i.e. $j(p)$ is called *the standard binary representation* of the *B*-bit number *p*.

PROPOSITION. It is easy to notice that function *y* from the set *P* of all *B*-bit numbers into 2^B defined in the following way:

(19) For all $p \in P$ $y(p) = A$ iff $p = (\bigcap_{a \in A} a) \cap (\bigcap_{b \in B \setminus A} \mathcal{U} \setminus b)$

is also a bijection.

As a result, for a given $p \in P$ we can use either standard binary representation $j(p)$, or set-like binary representation $y(p)$. The set *P* of *B*-bit numbers is well ordered by relation *Q* defined in (8). Bijections *j* and *y* transfer *Q* into the both 2^B and rng *j*.

3. NUMBERS

We have not been using numbers so far. The set *P* is called *B*-bit numbers, but it is an arbitrary name. We could another name, e.g. *B*-bit pieces of $\mathcal{U}$. The set *D* from definition 5, called digits, had elements $\circ$ (small circle), and $\mathbb{I}$ (vertical bar). Now, it is time to introduce numbers.

(20) Let us take $B = \{ \mathrm{b} \}$. In this particular case:

(21) $P = \{ \mathcal{U} \setminus b, b \}$

(22) $N = \{ \circ, \mathbb{I} \}$, ordered $< \circ, \mathbb{I} >$

(23) $2^B = \{ \emptyset, \{ b \} \}$

We can now use numbers $\circ$ and $\mathbb{I}$ obtained as a result of (20) to (23).

(24) Let us take $B_{\mathbb{I}} = \{ b_\circ, b_{\mathbb{I}} \}$ and make the reverse order $< b_{\mathbb{I}}, b_\circ >$

(25) $P_{\mathbb{I}} = \{ (\mathcal{U} \setminus b_{\mathbb{I}}) \cap (\mathcal{U} \setminus b_\circ), (\mathcal{U} \setminus b_{\mathbb{I}}) \cap b_\circ, b_{\mathbb{I}} \cap (\mathcal{U} \setminus b_\circ), b_{\mathbb{I}} \cap b_\circ \}$

(26) $N_{\mathbb{I}} = \{ \circ\circ, \circ\mathbb{I}, \mathbb{I}\circ, \mathbb{I}\mathbb{I} \}$ ordered $< \circ\circ, \circ\mathbb{I}, \mathbb{I}\circ, \mathbb{I}\mathbb{I} >$

(27) $2^{B_{\square}} = \{ \emptyset, \{ b_{\circ} \}, \{ b_{\square} \}, \{ b_{\circ}, b_{\square} \} \}$

We can now use numbers $\circ\circ, \circ\square, \square\circ, \square\square$ obtained as a result of (24) to (27).

(28) Let us take $B_{\square\square} = \{ b_{\circ\circ}, b_{\circ\square}, b_{\square\circ}, b_{\square\square} \}$, and make the reverse order
$< b_{\square\square}, b_{\square\circ}, b_{\circ\square}, b_{\circ\circ} >$

(29) $P_{\square\square} = \{ (\mathcal{U} \setminus b_{\square\square}) \cap (\mathcal{U} \setminus b_{\square\circ}) \cap (\mathcal{U} \setminus b_{\circ\square}) \cap (\mathcal{U} \setminus b_{\circ\circ}), ... \}$

(30) $N_{\square\square} = \{ \circ\circ\circ\circ, \circ\circ\circ\square, \circ\circ\square\circ, \circ\circ\square\square, \circ\square\circ\circ, \circ\square\circ\square, \circ\square\square\circ, \circ\square\square\square, \square\circ\circ\circ, ... \}$

(31) $2^{B_{\square\square}} = \{ \emptyset, \{ b_{\circ\circ} \}, \{ b_{\circ\square} \}, \{ b_{\circ\circ}, b_{\circ\square} \}, \{ b_{\square\circ} \}, \{ b_{\circ\circ}, b_{\square\circ} \}, \{ b_{\circ\square}, b_{\square\circ} \}, \{ b_{\circ\circ}, b_{\circ\square}, b_{\square\circ} \}, ... \}$

Having defined numbers we can now discuss a difference between the standard binary representation and the set-like binary representation. Any given number $n \in \mathbf{N}$ can be written as:

(32) $n = \sum_{i=0}^{K} c_i 2^i$

As we can see it is not a unique notation - it will become unique if we remove leading zeros. The set-like binary representation states:

(33) $\forall n \in \mathbf{N} \;\; \exists A \subset 2^{\mathbf{N}}$ such that $n = \sum_{i \in A} 2^i$

And, there exists one and only one set *A* meeting (33).

4. M-PLACED OPERATIONS ON BITS

Let $D = \{ 0, 1 \}$ be a set of binary digits. We now look at *m*-placed operations on *D*, but from a different perspective. Let us begin our considerations from one-placed operations, i.e. we shall have a look at:

(34) $O_1 = \{ f : D \to D$ - one-placed operation on $D \}$

DEFINITION 7. We now define the functor $\mathcal{F}_1$ from O_1 to 2^{B_1}. Let function $f \in O_1$ and set $A \in 2^{B_1}$ be given. Please note that B_1 was defined in (24).

(35) $\mathcal{F}_1(f) = A$ iff $\forall x \in D \;\; (f(x) = 1 \;\Leftrightarrow\; b_x \in A)$

PROPOSITION. The functor $\mathcal{F}_1$ from definition 7 is a bijection.

DEFINITION 8.

(36) $\forall f \in O_1 \;\; \exists ! F \in 2^{B_1}$ such that $F = \mathcal{F}_1(f)$ called *set regarded as function f.*

(37) $\forall F \in 2^{B_1} \;\; \exists ! f \in O_1$ such that $f = \mathcal{F}_1^{-1}(F)$ called *function defined by set F.*

Cardinality of 2^{B_1} equals 4. Therefore, there are four one-placed operations on *D* defined by elements of 2^{B_1}. They are:

(38) $\emptyset$ - equivalent of the logical operation false.

(39) $\{ b_0 \}$ - equivalent of the logical operation negation.

(40) $\{ b_1 \}$ - equivalent of the logical operation id_D.

(41) $\{ b_0, b_1 \}$ - equivalent of the logical operation true.

SAMPLES. $\emptyset(0) = 0, \emptyset(1) = 0$
$\{ b_0 \}(0) = 1, \{ b_0 \}(1) = 0$
$\{ b_1 \}(0) = 0, \{ b_1 \}(1) = 1$
$\{ b_0, b_1 \}(0) = 1, \{ b_0, b_1 \}(1) = 1$

Suppose we have two one-placed operations on *D*. We shall consider compositions of the operations. Let us denote by *f* the function defined by set *F* from 2^{B_1}, and by *g* the function defined by set *G* from 2^{B_1}. In other words: $f:D \to D$ and $g:D \to D$. As a result of (35) we can also write $F:D \to D$ and $G:D \to D$. We say that $H:D \to D$ is a composition of *F* and *G*, and denote by $H = F \bullet G$ iff

(42) $\forall x \in D \;\; H(x) = F \bullet G(x) = G(F(x))$

PROPOSITIONS. The following proposition are true:

(43) $\forall F \in 2^{B_1} \;\; F \bullet \emptyset = \emptyset$

(44) $\forall F \in 2^{B_1} \;\; F \bullet \{ b_0 \} = B_1 \setminus F$

(45) $\forall F \in 2^{B_1} \;\; F \bullet \{ b_1 \} = F$

(46) $\forall F \in 2^{B_1} \;\; F \bullet B_1 = B_1$

(47) $\forall G \in 2^{B_1} \;\; \emptyset \bullet G = \{ b_x : b_0 \in G \}$

(48) $\forall G \in 2^{B_1} \;\; \{ b_0 \} \bullet G = \{ b_x : b_{x'} \in G \}$, where *x'* denotes $\{ b_0 \}(x)$

(49) $\forall G \in 2^{B_1} \;\; \{ b_1 \} \bullet G = G$

(50) $\forall G \in 2^{B_1} \;\; B_1 \bullet G = \{ b_x : b_1 \in G \}$

Two-placed operations on *D* yield $[:D,D:]$ into *D*. By O_2 we denote the set of all two -placed operations on *D*.

(51) $O_2 = \{ f : [:D,D:] \to D$ - two-placed operations on $D \}$

DEFINITION 9. We now define functor $\mathcal{F}_2$ from O_2 to $2^{B_{11}}$. Let function $f \in O_2$ and set $F \in 2^{B_{11}}$ be given. Please note that B_{11} was defined in (28).

(52) $\mathcal{F}_2(f) = F$ iff $\forall x \in D \;\; (f(x,y) = 1 \Leftrightarrow b_{xy} \in F)$

PROPOSITION. The functor $\mathcal{F}_2$ from the definition 9 is a bijection.

DEFINITION 10.

(53) $\forall f \in O_2 \;\; \exists ! F \in 2^{B_{11}}$ such that $F = \mathcal{F}_2(f)$ called *set regarded as function f.*

(54) $\forall F \in 2^{B_{11}} \;\; \exists ! f \in O_2$ such that $f = \mathcal{F}_2^{-1}(F)$ called *function defined by set F.*

Cardinality of $2^{B_{11}}$ equals 16. Therefore, there are 16 two-placed operations on *D* defined by elements of $2^{B_{11}}$. We find amongst them:

(55) $\{ b_{11} \}$ - equivalent of conjuction;

(56) $\{ b_{01}, b_{10}, b_{11} \}$ - equivalent of disjunction;

(57) $\{ b_{01}, b_{10} \}$ - equivalent of exclusive or;

(58) $\{ b_{00}, b_{10}, b_{11} \}$ - equivalent of implication;

SAMPLE. $\{ b_{00}, b_{11} \}(0,0) = 1$, $\{ b_{00}, b_{11} \}(0,1) = 0$, $\{ b_{00}, b_{11} \}(1,0) = 0$, $\{ b_{00}, b_{11} \}(1,1) = 1$

Suppose we have two-placed operation *F* on *D*, and one-placed operation *G* on *D*. Composition $H = F \bullet G$ where:

(59) $\forall x \in D \;\; \forall y \in D \;\; H(x,y) = F \bullet G(x) = G(F(x,y))$

is a two-placed operation on *D*.

PROPOSITIONS. The following proposition are true:

(60) $\forall F \in 2^{B_{11}} \;\; F \bullet \emptyset = \emptyset$

(61) $\forall F \in 2^{B_{11}} \;\; F \bullet \{ b_0 \} = B_{11} \setminus F$

(62) $\forall F \in 2^{B_{11}}\ F \bullet \{ b_1 \} = F$

(63) $\forall F \in 2^{B_{11}}\ F \bullet B_{11} = B_{11}$

Suppose we have two one-placed operation *F* and *G* on *D*, and one two-placed operation *H* on *D*. Their composition $< F, G > \bullet H$ where

(64) $\forall x \in D\ \forall y \in D\ H(x,y) = < F, G > \bullet H(x,y) = H(F(x), G(y))$

is a two-placed operation on *D*.

PROPOSITIONS. Firstly, we assume that $G = \{ b_1 \} = \mathrm{id}_D$

(65) $\forall H \in 2^{B_{11}}\ < \emptyset, \mathrm{id}_D > \bullet H = \{ b_{xy} : b_{0y} \in H \}$

(66) $\forall H \in 2^{B_{11}}\ < \{ b_0 \}, \mathrm{id}_D > \bullet H = \{ b_{xy} : b_{x'y} \in H \}$

(67) $\forall H \in 2^{B_{11}}\ < \{ b_1 \}, \mathrm{id}_D > \bullet H = H$

(68) $\forall H \in 2^{B_{11}}\ < B_1, \mathrm{id}_D > \bullet H = \{ b_{xy} : b_{1y} \in H \}$

Secondly, we assume that $F = \{ b_1 \} = \mathrm{id}_D$

(69) $\forall H \in 2^{B_{11}}\ < \mathrm{id}_D, \emptyset > \bullet H = \{ b_{xy} : b_{x0} \in H \}$

(70) $\forall H \in 2^{B_{11}}\ < \mathrm{id}_D, \{ b_0 \} > \bullet H = \{ b_{xy} : b_{xy'} \in H \}$

(71) $\forall H \in 2^{B_{11}}\ < \mathrm{id}_D, \{ b_1 \} > \bullet H = H$

(72) $\forall H \in 2^{B_{11}}\ < \mathrm{id}_D, B_1 > \bullet H = \{ b_{xy} : b_{x1} \in H \}$

Suppose we have three two-placed operation *F*, *G* and *H* on *D*, i.e. from $[: D, D :]$ to *D*. Their composition $< F, G > \bullet H$ where

(73) $\forall x \in D\ \forall y \in D\ < F, G > \bullet H(x,y) = H(F(x,y), G(x,y))$

is a two-placed operation on *D*.

PROPOSITION. We shall prove the following proposition:

(74) $\forall H \in 2^{B_{11}} \setminus \{ \emptyset, B_{11} \}\ \forall A \in 2^{B_{11}}\ \exists F \in 2^{B_{11}}\ \exists G \in 2^{B_{11}}\ < F, G > \bullet H = A$

PROOF. Let $x \in D$ and $y \in D$ be given. Because $H \in 2^{B_{11}} \setminus \{ \emptyset, B_{11} \}$ we can state:

(75) $\exists i \in D$ and $\exists j \in D$ such that $b_{ij} \in H$

(76) $\exists l \in D$ and $\exists k \in D$ such that $b_{lk} \notin H$

There are two options:
1^0 $b_{xy} \in A$ then we take $F(x,y) = i$ and $G(x,y) = j$
2^0 $b_{xy} \notin A$ then we take $F(x,y) = l$ and $G(x,y) = k$

5. BITWISE OPERATIONS ON NUMBERS

Let us consider a collection of bits $< B, R >$. Suppose $F \in O_2$ and $G \in O_2$, i.e. *F* and *G* are two-placed operations on *D*. We now define operation on *B*-bit numbers $w : 2^B \to 2^B$ as follows:

DEFINITION 11. Let $X \in 2^B$ be given. To find $Y \in 2^B$ such that $w(X) = Y$ we perform the following steps:

(77) $c := 0 \in D$

(78) $\exists b \in B\ \forall a \in B \setminus \{ b \}$ holds $< a, b > \notin R$

(79) If $b \in X$ then $x := 1 \in D$. If $b \notin X$ then $x := 0 \in D$.

(80) $b \in Y \Leftrightarrow F(x,c)=1$

(81) $c := G(x,c)$

(82) $B := B \setminus \{b\}$

Then the step from (79) to (82) must be repeated until $B=\emptyset$. The above-constucted function w is called one-placed *bitwise operation on B-bit numbers*.

SAMPLE. If $G=\emptyset \in 2^{B_{11}}$ and $F=\{b_{00}\} \in 2^{B_{11}}$ then w is the bitwise negation.

Let us consider a collection of bits $<B,R>$. Suppose $F \in O_3$ and $G \in O_3$, i.e. F and G are three-placed operations on D. We now define operation on B-bit numbers $w: [\, 2^B, 2^B \,] \to 2^B$ as follows:

DEFINITION 12. Let $X \in 2^B$ and $Y \in 2^B$ be given. To find $Z \in 2^B$ such that $w(X,Y)=Z$ we perform the following steps:

(83) $c := 0 \in D$

(84) $\exists b \in B \ \forall a \in B \setminus \{b\}$ holds $<a,b> \notin R$

(85) If $b \in X$ then $x := 1 \in D$. If $b \notin X$ then $x := 0 \in D$.

(86) If $b \in Y$ then $y := 1 \in D$. If $b \notin Y$ then $y := 0 \in D$.

(87) $b \in Z \Leftrightarrow F(x,y,c)=1$

(88) $c := G(x,y,c)$

(89) $B := B \setminus \{b\}$

Then the step from (84) to (89) must be repeated until $B=\emptyset$. The above-constucted function w is called two-placed *bitwise operation on B-bit numbers*.

SAMPLES.
If $G=\emptyset \in 2^{B_{111}}$ and $F=\{b_{110}\} \in 2^{B_{111}}$ then w is the bitwise conjunction.

If $G=\emptyset \in 2^{B_{111}}$ and $F=\{b_{000}, b_{010}, b_{100}\} \in 2^{B_{111}}$ then w is the bitwise disjunction.

If $G=\emptyset \in 2^{B_{111}}$ and $F=\{b_{010}, b_{100}\} \in 2^{B_{111}}$ then w is the bitwise exclusive or.

If $F=\{b_{001}, b_{010}, b_{100}, b_{111}\} \in 2^{B_{111}}$ and $G=\{b_{011}, b_{101}, b_{110}, b_{111}\} \in 2^{B_{111}}$
then w is the addition of B-bit numbers.

If $F=\{b_{001}, b_{010}, b_{100}, b_{111}\} \in 2^{B_{111}}$ and $G=\{b_{001}, b_{010}, b_{011}, b_{111}\} \in 2^{B_{111}}$
then w is the subtraction of B-bit numbers.

CONCLUSION. Addition of B-bit numbers is a bitwise operation defined by a tuple of two 4-element sets. If we convert binary index into decimal numbers and use index instead of b_n we shall obtain

(90) $<\{1,2,4,7\}, \{3,5,6,7\}>$

as unique representation of the addition of B-bit numbers.

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